Asymptotic expansions of a manifold near its curve of singular points

Alexander Bruno and Alijon Azimov

Abstract. In [1–3] parametric expansions near 5 singular points and 3 curves consisting of singular points were computed for a two-dimensional algebraic manifold Ω. Here we present general methods for computing the expansions of a manifold near its curve of singular points and their application to a single curve $\mathcal{F}$.

1. Introduction

In [4–8] the study of the three-parameter family of special homogeneous spaces in terms of the normalized Ricci flow was started. Ricci flows give the evolution of Einstein metrics on a manifold. The equation of the normalized Ricci flow reduces to a system of two ordinary differential equations with three parameters a_1,a_2 and a_3:

$$\begin{aligned} \frac{dx_1}{dt} &= \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \\ \frac{dx_2}{dt} &= \tilde{f}_2(x_1, x_2, a_1, a_2, a_3), \end{aligned} \tag{1}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are some concrete functions.

The singular points of this system correspond to Einstein invariant metrics. At a singular (fixed) point x_1^0, x_2^0 the system (1) has two eigenvalues λ_1 and λ_2. If at least one of them is equal to zero, the singular point x_1^0, x_2^0 is called degenerate. In [4–8] a theorem is proved that the set Ω of values of parameters a_1, a_2, a_3, at which the system (1) has at least one degenerate singular point is described by the equation

$$\begin{aligned}Q(s_1,s_2,s_3)\overset{\text{def}}{\equiv}&(2s_1+4s_3-1)\left(64s_1^5-64s_1^4+8s_1^3+240s_1^2s_3-1536s_1s_3^2-\right.\\&\left.-4096s_3^3+12s_1^2-240s_1s_3+768s_3^2-6s_1+60s_3+1\right)-\\&-8s_1s_2(2s_1+4s_3-1)(2s_1-32s_3-1)(10s_1+32s_3-5)-\\&-16s_1^2s_2^2\left(52s_1^2+640s_1s_3+1024s_3^2-52s_1-320s_3+13\right)+\\&+64(2s_1-1)s_2^3(2s_1-32s_3-1)+2048s_1(2s_1-1)s_2^4=0,\end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal, respectively, to

$$s_1=a_1+a_2+a_3,\quad s_2=a_1a_2+a_1a_3+a_2a_3,\quad s_3=a_1a_2a_3.$$

In [9] for symmetry reasons, from coordinates $\mathbf{a}=(a_1,a_2,a_3)$ authors passed to the coordinates $\mathbf{A}=(A_1,A_2,A_3)$ by linear substitution

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=M\cdot\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix},\quad M=\begin{pmatrix}(1+\sqrt{3})/6 & (1-\sqrt{3})/6 & 1/3\\(1-\sqrt{3})/6 & (1+\sqrt{3})/6 & 1/3\\-1/3 & -1/3 & 1/3\end{pmatrix}$$

Definition 1. *Let $\varphi(X)$ be some polynomial, $X=(x_1,\ldots,x_n)$. Point $X=X^0$ of the set $\varphi(X)=0$ is called a* ***singular point of*** k***-order****, if in this point all partial derivatives of the polynomial $\varphi(X)$ by $x_1,\ldots,x_n$ go to zero up to k-th order and at least one partial derivative of order $k+1$ does not go to zero.*

In [9] all singular points of the manifold Ω were found in coordinates $\mathbf{A}=(A_1,A_2,A_3)$. Five third-order points,

Name	Coordinates A
$P_1^{(3)}$	$(0,0,3/4)$
$P_2^{(3)}$	$(0,0,-3/2)$
$P_3^{(3)}$	$\left(-\frac{1+\sqrt{3}}{2},\frac{\sqrt{3}-1}{2},\frac{1}{2}\right)$
$P_4^{(3)}$	$\left(\frac{\sqrt{3}-1}{2},-\frac{1+\sqrt{3}}{2},\frac{1}{2}\right)$
$P_5^{(3)}$	$(1,1,1/2)$

three second-order points,

Name	Coordinates A
$P_1^{(2)}$	$\left(\frac{1+\sqrt{3}}{4},\frac{1-\sqrt{3}}{4},\frac{1}{2}\right)$
$P_2^{(2)}$	$\left(\frac{1-\sqrt{3}}{4},\frac{1+\sqrt{3}}{4},\frac{1}{2}\right)$
$P_2^{(3)}$	$(1,1,1/2)$

and three more algebraic curves of singular points of first order

$$\mathcal{F} = \left\{a_1 = a_2, 16a_1^3 + 16a_1^2 a_3 - 4a_1 - 2a_3 + 1 = 0\right\},$$

$$\mathcal{I} = \left\{A_1 + A_2 + 1 = 0, A_3 = \frac{1}{2}\right\},$$

$$\mathcal{K} = \left\{A_1 = -\frac{9}{4} th\,(t)\,,\ A_2 = -\frac{9}{4} h\,(t)\,,\ A_3 = \frac{3}{4},\ h\,(t) = \frac{t^2 + 1}{(t+1)(t^2 - 4t + 1)}\right\}.$$

In this case, the points $P_3^{(3)}$, $P_4^{(3)}$ and $P_5^{(3)}$ are of the same type, they pass into each other at rotation around the origin of the plane A_1, A_2 by an angle $2\pi/3$, just as all points $P_1^{(2)}$, $P_2^{(2)}$, $P_3^{(2)}$. The curves $\mathcal{F}$, $\mathcal{I}$, $\mathcal{K}$ correspond to two more curves of the same type. Therefore, it is enough to study the manifold Ω in the neighborhoods of the points $P_1^{(3)}$, $P_2^{(3)}$, $P_5^{(3)}$, $P_3^{(2)}$ and curves $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{K}$. Moreover, in [9] the cross sections of the manifold Ω by the planes $A_3 = \text{const}$, were calculated and it was shown that in a finite part of the space $\mathbb{R}^3 = \{A_1, A_2, A_3\}$ the manifold Ω consists of one-dimensional branches F_1, F_2, F_3, and two-dimensional branches G_1, G_2, G_3 which are broken into parts $F_i^{\pm}$, $G_i^{\pm}$ with boundaries at the plane $A_3 = 1/2$.

The structure of the manifold Ω near singular points $P_i^{(3)}$ and $P_i^{(2)}$ was considered in [1,2]. The structure of the manifold Ω near three algebraic curves $\mathcal{I}$, $\mathcal{K}$, $\mathcal{F}$ of singular points of the first order was considered in [3]. For this study, we use the following algorithm consisting of 8 steps.

2. Calculation scheme

Step 1: Introduce local coordinates $X = (x_1, x_2, x_3)$. If we consider a straight line consisting of singular points (as $\mathcal{I}$), then one coordinate x_1 is directed along the line and coordinates x_2,x_3 describe deviations from the line. If the curve is located on a plane, we introduce the coordinate x_3, normal to this plane, coordinates x_1,x_2 of the curve on the plane are parameterized $x_1 = b_1(t)$, $x_2 = b_2(t)$ and a coordinate $y_2 = x_2 - b(t)$ of the deviation from this curve.

Step 2: The original polynomial $R(\mathbf{A})$ write in local coordinates as

$$g(t, y_2, x_3) = \sum \varphi(t)_{pq} y_2^p x_3^q, \tag{2}$$

and compute its support $\mathbf{S} = \{(p, q) : \varphi_{pq} \not\equiv 0\}$. Let the support $\mathbf{S}$ consists of points (p_i, q_i), $i = 1, \ldots, k$.

Step 3: Newton's polygon $\Gamma(g)$ is calculated as a convex hull of the support $\mathbf{S}$:

$$\Gamma(g) = \left\{(p, q) = \sum_{i=1}^{k} \lambda_i (p_i, q_i),\ \lambda_i \geqslant 0,\ i = 1, \ldots, k,\ \sum_{i=1}^{k} \lambda_i = 1\right\}.$$

Boundary $\partial\Gamma$ of polygon $\Gamma(g)$ consists of its vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$ which we call as generalized faces. Here j is the number of the generalized face $\Gamma_j^{(d)}$. Each face $\Gamma_j^{(d)}$ corresponds to its truncated polynomial

$$\hat{g}_j^{(d)}(Y) = \sum g_{(p,q)} y_2^p x_3^q \text{ over } (p,q) \in \mathbf{S} \cap \Gamma_j^{(d)}$$

and the normal cone $\mathbf{U}_j^{(d)}$, consisting of all normals to the face $\Gamma_j^{(d)}$, which are the external normals to the polygon Γ. For their computation we use PolyhedralSets of the computer algebra system (CAS) Maple package [10].

Step 4: Select the faces $\Gamma_j^{(1)}$ with normals $N_j \leqslant 0$ and corresponding truncated polynomials $\hat{g}_j^{(1)}(t, y_2, x_3)$.

Step 5: For each selected truncated polynomial $\hat{g}_j^{(1)}(t, y_2, x_3)$, we calculate the corresponding power transformations

$$(\ln y_2, \ln x_3) = (\ln z_1, \ln z_3)\, \alpha, \tag{3}$$

where α is such a unimodular matrix 2×2, that

$$\hat{g}_j^{(1)}(t, y_2, x_3) = h(z_1, t) z_3^l \tag{4}$$

with a multiplier z_3^l.

Step 6: We make the power transformation (3) in the polynomial (2) itself and write it in the following form

$$g(Z) = T(z_1, t, z_3) = z_3^l \sum_{k=0}^{m} T_k(z_1, t) z_3^k,$$

with some natural number m. The polynomial $T_k(z_1, t)$ is calculated by the command `coeff(T,z[k],m)` in CAS `Maple`, and $T_0(z_1, t) = h(z_1, t)$ from Equality (4).

Step 7: If $T_0(z_1(t), t) \not\equiv 0$, then we substitute in the polynomial $T(z_1, t, z_3) z_3^{-l}$

$$z_1 = b_1(t) + \varepsilon, \quad z_2 = b_2(t) + \varepsilon \tag{5}$$

and obtain the function $u(\varepsilon, t, z_3) = T(z_1, z_2, z_3) z_3^{-l}$. Now we apply to the equation $u(\varepsilon, t, z_3) = 0$ Theorem 1 [1] on the generalized implicit function and obtain the parametric expansion

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) z_3^k. \tag{6}$$

Step 8: Calculate several terms of expansion (6) and substitute them into (5). The result is substituted into the power transformation (3) and we obtain the parametric expansion of Ω into a power series by z_3, with coefficients which are rational functions of the t.

3. Structure of the manifold Ω near the Curve $\mathcal{F}$ of singular points

Theorem 1. *The curve $\mathcal{F}$ consists of branches $F_1^{\pm}$,$F_2^{\pm}$,$F_3^{\pm}$. On them two-dimensional branches $G_1^{\pm}$,$G_2^{\pm}$,$G_3^{\pm}$ of the manifold Ω meet (but do not intersect).*

References

[1] A.D. Bruno and A.A. Azimov. *Parametric expansions of an algebraic variety near its singularities.* Axioms 2023, p. 469. https://doi.org/10.3390/axioms12050469.

[2] A.D. Bruno and A.A. Azimov. *Parametric expansions of an algebraic variety near its singularities.* International Conference on Polynomial Computer Algebra St.Petersburg, April 17-22, 2023. page 22–26. ISSBN 978-5-9651-1473-3.

[3] A.D. Bruno and A.A. Azimov. *Parametric expansions of an algebraic variety near its singularities II.* Axioms 2024, 13(2), 106. https://doi.org/10.3390/axioms13020106.

[4] A.L. Besse. *Einstein Manifolds*, Springer: Berlin, Germany, 1987.

[5] N.A. Abiev, A. Arvanitoyeorgos, Y.G. Nikonorov and P. Siasos. *The dynamics of the Ricci flow on generalized Wallach spaces*, *Differ. Geom. Its Appl.* 2014, *35*, 26–43. https://doi.org/10.1016/j.difgeo.2014.02.002.

[6] N.A. Abiev, A. Arvanitoyeorgos, Y.G. Nikonorov and P. Siasos. *The normalized Ricci flow on generalized Wallach spaces*, In *Mathematical Forum*; Studies in Mathematical Analysis; Yuzhnii Matematicheskii Institut, Vladikavkazskii Nauchnii Tsentr Rossiyskoy Akademii Nauk: Vladikavkaz, Russia, 2014; Volume 8, pp. 25–42. (In Russian)

[7] N.A. Abiev and Y.G. Nikonorov. *The evolution of positively curved invariant Riemannian metrics on the Wallach spaces under the Ricci flow. Ann. Glob. Anal. Geom.* 2016, *50*, 65–84. https://doi.org/10.1007/s10455-016-9502-8.

[8] M. Jablonski. *Homogeneous Einstein manifolds.*, *Rev. Unión Matemática Argent.* 2023, *64*, 461–485. https://doi.org/10.33044/revuma.3588.

[9] A.D. Bruno and A.B. Batkhin. *Investigation of a real algebraic surface.. Program. Comput. Softw.* 2015, *41*, 74–82. https://doi.org/10.1134/S0361768815020036.

[10] I.Thompson. *Understanding Maple*, Cambridge University Press: Cambridge, UK, 2016.

Alexander Bruno
Singular Problems Department
Keldysh Institute of Applied Mathematics of RAS
Moscow, Russia
e-mail: abruno@keldysh.ru

Alijon Azimov
Algebra and Geometry Department
Samarkand State University after Sh. Rashidov
Samarkand, Uzbekistan
e-mail: Azimov_Alijon_Akhmadovich@mail.ru